

BANGALORE UNIVERSITY
DEPARTMENT OF MATHEMATICS
CENTRAL COLLEGE CAMPUS
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CENTRAL COLLEGE MATHEMATICAL SOCIETY

LECTURE NOTICE

Prof. S.K. LAKSHMANA RAO

Professor Emeritus,

Department of Mathematics, Regional Engineering College, Warangal.

has kindly consented to give a talk on

“INTRODUCTION TO WAVELETS”

on Thursday 18TH November 1999 at 3.00 p.m.

Venue : Mathematics Big Hall, Central College, Bangalore - 1

All are cordially invited to attend.

K. S. Harinath
K.S. HARINATH 12/11/99
CHAIRMAN

FOURIER ANALYSIS ; WAVELET ANALYSIS

The differential equation of the vibrating spring :

$$y'' + \alpha y' + \kappa^2 y = p(t) \quad (1)$$

($m = \text{mass}$ ^{of the} spring is normalized ; $\alpha = \text{damp parameter}$; $\kappa^2 = \text{spring constant}$; $p(t) = \text{exciting force}$). Eqn. (1) has two important features: (i) Linearity , (ii) characteristic parameters are constant (i.e. α , and κ^2 are constant).

Taking $p(t) = A \exp(i\omega t)$, the solution (after ignoring the transient components springing from the complementary function) is

$$y(t) = B \exp(i\omega t) \quad (2)$$

and

$$B = \frac{A}{\kappa^2 - \omega^2 + i\alpha\omega} = c \exp(-i\phi) \quad (3)$$

$$c = A \{ (\kappa^2 - \omega^2)^2 + \alpha^2 \omega^2 \}^{-1/2} , \quad \tan \phi = \frac{\alpha \omega}{\kappa^2 - \omega^2} \quad (4)$$

(2)

The output to a sinusoidal input is a sinusoid with the same frequency ω but the amplitude and phase are both modified (c.f. eqn. (4)).

Why should the exciting force $p(t)$ be of the particular type chosen above? An answer to this question highlights the vital role of the Fourier Series.

Fourier's theorem asserts that an 'arbitrary' function $p(t)$ can be resolved into pure sine and cosine vibrations in the form

$$p(t) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) \quad (5)$$

if the variable t has the finite range $[0, T]$ with $T = \frac{2\pi}{\omega}$. The expansion

(5) is also representable in the form

$$p(t) = \sum_{-\infty}^{\infty} c_n \exp(i n \omega t) \quad (6)$$

with

$$c_0 = \frac{1}{2} a_0, \quad c_n = \frac{1}{2} (a_n - i b_n), \quad c_{-n} = \bar{c}_n = \frac{1}{2} (a_n + i b_n) \quad (7)$$

and the output for an 'arbitrary' input $p(t)$ which is defined on the finite range $[0, T]$ is obtained by superposition of the outputs corresponding to $p_n(t) = c_n \exp(in\omega t)$.

The Fourier coefficients $\{a_n, b_n\}$ or $\{c_n\}$ are computed from the formulae:

$$\left. \begin{aligned} (a_n, b_n) &= \frac{2}{T} \int_0^T p(t) (\cos n\omega t, \sin n\omega t) dt \\ c_n &= \frac{1}{T} \int_0^T p(t) \exp(-in\omega t) dt. \end{aligned} \right\} (8)$$

The functions $\exp(in\omega t)$ $n = 0, 1, -1, 2, -2, \dots$ have the orthogonal property over $[0, T]$ which is directly verifiable. A way to prove this statement that an 'arbitrary' function $p(t)$ has indeed a representation of the form (5) or (6) is to regard the N th order Fourier polynomial

$$S_N(t) = \frac{1}{2} a_0 + \sum_{k=1}^N (a_k \cos k\omega t + b_k \sin k\omega t) \quad (9)$$

is an approximation to $p(t)$ for every

choice of $N =$ an arbitrary (but finite) positive

integer

(in the least square error sense) and to

show that under a set of suitable conditions

the "error" $\int_0^T |p(t) - s_N(t)|^2 dt$ has

the limit zero at all points $t \in [0, T]$

barring discontinuities of $p(t)$. The

Fourier coefficients $\{c_n\}$ (eqn. (8)) have

the feature of 'finiteness' and the formulae

determining them are independent of the

order N of the approximating polynomial

$s_N(t)$.

Dirichlet's conditions: Without loss of

generality we may take the basic interval of

interest to be $[-\pi, \pi]$. D's conditions

are that (i) f is single-valued in the

range $[-\pi, +\pi]$, (ii) f is bounded and

sectionally continuous, except for a finite number

of jumps and (iii) f has only a finite number of

maxima and minima in the range $[-\pi, +\pi]$.

convergence of the Fourier Series for the well circumscribed class of functions that satisfy Dirichlet conditions.

In the formal expansion of $f(x)$

$$f(x) = \frac{1}{2} a_0 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \quad (10)$$

$$(a_n, b_n) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) (\cos n\theta, \sin n\theta) d\theta. \quad \text{To}$$

achieve convergence we have to consider the approximation

$$f_N(x) = \frac{1}{2} a_0 + \sum_{k=1}^N (a_k \cos kx + b_k \sin kx) \quad (11)$$

and explore the limit as $N \rightarrow \infty$. Replacing

(a_k, b_k) in (11) using their values in integral

form, we get

$$f_N(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \left[\frac{1}{2} + \sum_{k=1}^N \cos k(\theta - x) \right] d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta) \frac{\sin(N + \frac{1}{2})(\theta - x)}{\sin \frac{\theta - x}{2}} d\theta \quad (12)$$

or using the trigonometric identity

$$D_N(t) = \frac{1}{2} + \cos t + \dots + \cos Nt = \frac{\sin(N + \frac{1}{2})t}{2 \sin \frac{t}{2}} \quad (13)$$

If we anticipate that $\lim_{N \rightarrow \infty} f_N(x) = f(x)$, we

see that the integral in (12) receives contribution only from the immediate neighbourhood of the point $\theta = x$ though the integral has the entire range $(-\pi, \pi)$.

$$f_N(x) = \int_{-\pi}^{x-\delta} + \int_{x-\delta}^{x+\delta} + \int_{x+\delta}^{\pi} \left[\frac{f(\theta) \sin(N+\frac{1}{2})(\theta-x)}{2\pi \sin \frac{\theta-x}{2}} \right] d\theta$$

$$= I_1 + I_2 + I_3 \quad (14)$$

In I_1 and I_3 the function $\frac{f(\theta)}{2\pi \sin \frac{\theta-x}{2}}$ is bounded and integrable (a consequence of Dirichlet's condition number 2) and $\sin(N+\frac{1}{2})(\theta-x)$ oscillates rapidly as N increases. We have the classic result

that if $\phi(x)$ is bounded and integrable in (α, β)

$$\int_{\alpha}^{\beta} \phi(x) \sin mx \, dx \text{ converges to zero as } m$$

increases. Thus to examine the limit of $f_N(x)$, we need consider only the integral

$$I_2 = \int_{x-\delta}^{x+\delta} \frac{f(\theta) \sin(N+\frac{1}{2})(\theta-x)}{2\pi \sin \frac{\theta-x}{2}} d\theta = \frac{1}{\pi} \int_{x-\delta}^{x+\delta} f(\theta) C_N(\theta-x) d\theta$$

$$= \frac{1}{\pi} \int_{-\delta}^{\delta} f(x+t) C_N(t) dt \quad (15)$$

in the interval $(x-\delta, x+\delta)$, $f(x)$ has at best a jump discontinuity at x and let $f(x_+)$ and $f(x_-)$ denote the right and left limits of $f(x)$.

$$I_2 = \frac{1}{\pi} \int_{-\delta}^{\delta} f(x+t) c_N(t) dt = \frac{1}{\pi} \int_{-\delta}^0 [f(x+t) - f(x_-)] c_N(t) dt + \frac{1}{\pi} \int_0^{\delta} [f(x+t) - f(x_+)] c_N(t) dt + \frac{1}{2\pi} \int_{-\delta}^{\delta} [f(x_-) + f(x_+)] c_N(t) dt$$

Introduce the function

$$g(t) = \begin{cases} f(x+t) - f(x_-) & \text{if } t \leq 0 \\ f(x+t) - f(x_+) & \text{if } t \geq 0 \end{cases} \quad (16)$$

$f(x)$ is continuous ~~to the left~~ or has a finite jump and $g(t)$ is continuous in the range $[-\delta, \delta]$.

Taking $f(x)$ to be a function of bounded variation

in $(x-\delta, x+\delta)$ we see that $g(t)$ has a

similar property in $(-\delta, \delta)$ and hence

$$g(t) = u(t) - v(t) \quad (17)$$

where both u and v are monotonic and positive.

$$\begin{aligned} \frac{1}{\pi} \int_{-\delta}^{\delta} g(t) c_N(t) dt &= \frac{1}{\pi} \int_{-\delta}^{\delta} [u(t) - v(t)] c_N(t) dt \\ &= \frac{1}{\pi} \left[u(-\delta) \int_{-\delta}^{n_1} c_N(t) dt - v(-\delta) \int_{-\delta}^{n_2} c_N(t) dt \right] \end{aligned}$$

(8)

$$\int_a^b c_N(t) dt = \left(\int_0^b - \int_0^a \right) \left[\frac{\sin(N+\frac{1}{2})\theta}{2 \sin \frac{\theta}{2}} d\theta \right]$$

$$\text{and } \int_0^\alpha \frac{\sin(N+\frac{1}{2})\theta}{2 \sin \frac{\theta}{2}} d\theta = \text{Si}(N+\frac{1}{2})\alpha + \epsilon_1(\alpha)$$

where $\epsilon_1(\alpha) \rightarrow 0$ as $N \rightarrow \infty$ and $\text{Si}(N+\frac{1}{2})\alpha \rightarrow \frac{\pi}{2}$

if $\alpha > 0$. It follows that

$$\frac{1}{\pi} \left| \int_{-\delta}^{\delta} g(t) c_N(t) dt \right| < [u(-\delta) + v(-\delta)] \times \text{a constant}$$

and both $u(-\delta)$, $v(-\delta)$ go to zero with decreasing δ .

$$\text{In the limit as } N \rightarrow \infty, I_2 \rightarrow \frac{1}{2\pi} (f(x_-) + f(x_+))\pi$$

$$= \frac{1}{2} f(x_-) + f(x_+)$$

thus confirming that the Fourier expansion converges to its generator function $f(x)$ at all points of continuity and that at jump discontinuities the series converges to the average of the two limits on the two sides.

Proceeding in a way similar to the above, we can show that the class of C^1 functions on $[-\pi, \pi]$ (with the existence of the first derivative) has convergent Fourier expansions.

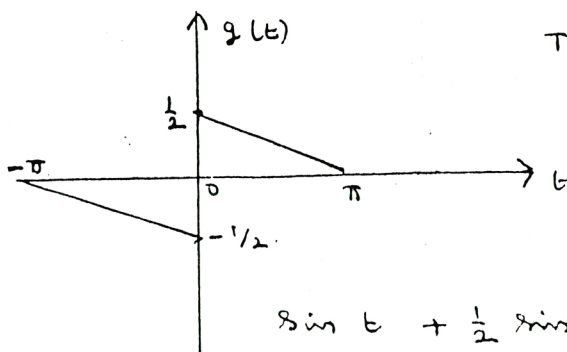
Gibbs Phenomenon

The Fourier Series of

a sectionally differentiable function is convergent and even uniformly convergent in sub-intervals that exclude points of discontinuity. A finite Fourier polynomial approximation gives rise to an error that can be made lower than a certain ϵ_N if there is no point of discontinuity in the interval. The error cannot be controlled by a mere increase of the order of approximation N if we need the evaluation in the immediate neighbourhood of a point of discontinuity.

Consider the 'saw tooth function'

$$g(t) = \frac{1}{2} - \frac{t}{2\pi} \quad (t > 0), \quad = 0 \quad (t = 0), \quad -\frac{1}{2} - \frac{t}{2\pi} \quad (t < 0).$$



The Fourier expansion of $g(t)$

is

$$\frac{1}{\pi} \sum_{k=1}^{\infty} \frac{\sin kt}{k} \quad (18)$$

$$\sin t + \frac{1}{2} \sin 2t + \dots + \frac{1}{n} \sin nt =$$

$$Si \left(n + \frac{1}{2} \right) t - \frac{1}{2} t + \frac{\alpha_n(t)}{n} \quad (19)$$

where $x_n(t)$ is bounded. Ignoring this term in (19),

we have from (18), (19) the difference

$$g(t) - g_n(t) = \frac{1}{2} - \frac{1}{\pi} \text{Si}\left(n + \frac{1}{2}\right)t = \eta_n(t) \quad (20)$$

and for large values of n (with an accuracy of n^{-2}),

$$\text{Si}\left(n + \frac{1}{2}\right)t = \frac{\pi}{2} - \frac{\cos\left(n + \frac{1}{2}\right)t}{\left(n + \frac{1}{2}\right)t} \quad \text{and} \quad (21)$$

$$\eta_n(t) = \frac{\cos\left(n + \frac{1}{2}\right)t}{\pi\left(n + \frac{1}{2}\right)t}$$

This converges to zero as $n \rightarrow \infty$ if $t \neq 0$. The

error depends only on $\left(n + \frac{1}{2}\right)t$. With increasing n

the error remains constantly the same if the scale

of t decreases. The error oscillations do not

diminish but are reduced to a gradually shrinking

range. For large n there is the picture of

rapid damped oscillations whose envelope starts

with the value 0.089490 and decreases

rapidly to zero. The maximum error (≈ 0.08949)

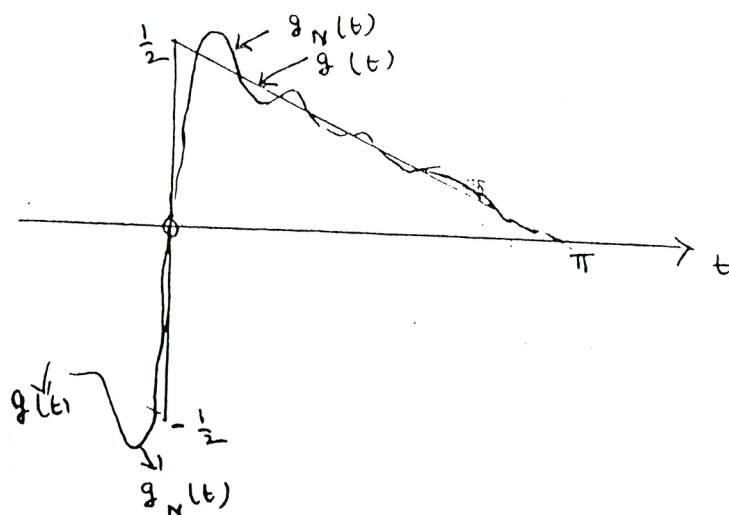
represents an overshoot of about 18% compared

with the jump $\approx \frac{1}{2}$ measured from the symmetry

point $t = 0$. Increase of n does not change

(11)

the picture of the overshoot but only telescopes the feature in the direction of the t -axis to a still smaller range. This is Gibbs' phenomenon.



Fejér's arithmetic mean method

$$f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx) \quad (2.2)$$

$$S_N(x) = \frac{1}{2} a_0 + \sum_{k=1}^N (a_k \cos kx + b_k \sin kx)$$

$$= f_N(x) = \int_{-\pi}^{\pi} f(\theta) D_N(\theta - x) d\theta \quad (2.3)$$

Let σ_N be the arithmetic mean $\frac{1}{N} (S_0 + S_1 + \dots + S_{N-1})$

$$\begin{aligned} &= \frac{1}{N} \sum_{k=0}^{N-1} f_k(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \left(\sum_{k=0}^{N-1} D_k(\theta - x) \right) d\theta \\ &= \frac{1}{2N\pi} \int_{-\pi}^{\pi} f(\theta) \left(\frac{\sin^2 \frac{N}{2}(\theta - x)}{\sin^2 \frac{\theta - x}{2}} \right) d\theta \quad (2.4) \end{aligned}$$

$$\left\{ \sin \frac{\theta}{2} + \sin \frac{3\theta}{2} + \dots + \sin \frac{2N-1}{2} \theta \right\} = \frac{\sin^2 \frac{N\theta}{2}}{\sin \frac{\theta}{2}} \quad \left. \vphantom{\frac{\sin^2 \frac{N\theta}{2}}{\sin \frac{\theta}{2}}} \right\}.$$

The kernel $M_N(\theta - x) = \frac{\sin^2 \frac{N}{2}(\theta - x)}{2N \sin^2 \frac{\theta - x}{2}}$ is

nowhere negative and has the factor $\frac{1}{2N}$. The

integral in (23) can be split into a sum of

three integrals over the ranges $(-\pi, x-\delta)$,

$(x-\delta, x+\delta)$ and $(x+\delta, \pi)$ as in (14) and

the first and last integrals approach zero as $N \rightarrow \infty$,

in view of the sign of $M_N(\theta - x)$ and the presence

of the term N in the denominator. The middle

integral is

$$\bar{I}_N(x) = \frac{2}{\pi} \int_{-\delta}^{\delta} f(x+t) \frac{\sin^2 \frac{Nt}{2}}{t^2} dt \quad (25)$$

after ignoring an additional integral which has

the limit zero as $N \rightarrow \infty$. We can write

$$\bar{I}_N(x) = \frac{f(x)}{\pi} \int_{-\delta}^{\delta} M_N(\theta) d\theta + \frac{1}{\pi} \int_{-\delta}^{\delta} g(\theta) M_N(\theta) d\theta \quad (26)$$

Since $M_N(\theta) > 0$, the second integral above is

$$\frac{1}{\pi} g(x\delta) \int_{-\delta}^{\delta} M_N(t) dt. \quad \text{Since } \int_{-\pi}^{\pi} M_N(t) dt = \pi$$

(12)

$$\lim_{N \rightarrow \infty} \int_{-\delta}^{\delta} M_N(t) dt = \pi \quad \text{and} \quad g(t) \text{ is}$$

continuous around $t=0$ and $g(0)=0$. Hence

$$\frac{1}{\pi} \int_{-\delta}^{\delta} g(\theta) M_N(\theta) d\theta \text{ is negligibly small for}$$

sufficiently small δ and

$$\lim_{N \rightarrow \infty} \bar{f}_N(x) = \lim_{N \rightarrow \infty} \bar{I}_N(x) = f(x). \quad (27)$$

The arithmetic mean of the partial sums to $f(x)$ converges to $f(x)$ at all points as long as $f(x)$ is integrable and has at most a finite number of finite jumps in the interval $(-\pi, \pi)$.

Additional remarks: The integral (25)

has the property

$$m \leq \bar{I}_N(x) \leq M \quad (28)$$

where m and M are the minimum and maximum values of $f(x)$ in the range $[x-\delta, x+\delta]$ and

for large N the integral (28) and $\bar{f}_N(x)$

come ^{arbitrarily} close to each other. If $f(x) - \bar{f}_N(x) = \bar{\eta}_N(x)$,

from (28) we see that

$$|\bar{\eta}_N(x)| \leq M - m \quad (N \rightarrow \infty). \quad (29)$$

Divide $[-\pi, \pi]$ into a large number of sub-intervals of length δ and allow δ to arbitrarily small values. By definition of Riemann integration,

$$\int_{-\pi}^{\pi} |\bar{f}_N(x)|^2 dx < \delta \sum (M-m)^2 \quad (N \rightarrow \infty).$$

If $f(x)$ is square-integrable, $\int_{-\pi}^{\pi} [f(x)]^2 dx < \infty$

and
$$\lim_{N \rightarrow \infty} \int_{-\pi}^{\pi} [f(x) - \bar{f}_N(x)]^2 dx = 0. \quad (30)$$

$$\bar{f}_N(x) = \frac{1}{N} \sum_{k=0}^{N-1} f_k(x) = \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) (a_k \cos kx + b_k \sin kx)$$

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \bar{f}_N(x) dx &= \pi \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right) (a_k^2 + b_k^2) \\ \int_{-\pi}^{\pi} (\bar{f}_N(x))^2 dx &= \pi \sum_{k=0}^{N-1} \left(1 - \frac{k}{N}\right)^2 (a_k^2 + b_k^2) \end{aligned}$$

Let
$$\int_{-\pi}^{\pi} [f(x)]^2 dx = A$$

$$\int_{-\pi}^{\pi} [f(x) - \bar{f}_N(x)]^2 dx = A - \pi \sum_{k=0}^{N-1} \left\{ 2 \left(1 - \frac{k}{N}\right) - \left(1 - \frac{k}{N}\right)^2 \right\} (a_k^2 + b_k^2)$$

and using (30) we see that as $N \rightarrow \infty$,

$$\pi \sum_{k=0}^{N-1} (a_k^2 + b_k^2) = A + \pi \sum_{k=1}^N \frac{k^2}{N^2} (a_k^2 + b_k^2).$$

The sum on the left side is less than or equal to A

for all N (Bessel's inequality) and hence

$$\lim_{N \rightarrow \infty} \frac{1}{N^2} \sum_{k=1}^N k^2 (a_k^2 + b_k^2) = 0. \quad (31)$$

Hence the completeness relation:

$$\lim_{N \rightarrow \infty} \left(\pi \sum_{k=0}^{N-1} (a_k^2 + b_k^2) \right) = A = \int_{-\pi}^{\pi} [f(x)]^2 dx. \quad (32)$$

$$\text{Thus } \lim_{N \rightarrow \infty} \int_{-\pi}^{\pi} [f(x) - f_N(x)]^2 dx = 0 \quad (33)$$

for all integrable and square-integrable functions

even if $r_N(x)$ may not converge to zero at any point.

Fejér's arithmetic mean method gives better convergence than the Dirichlet sum. Gibb's oscillations in the neighbourhood of a discontinuity are eliminated in Fejér's method.

Fourier Integral | Fourier Transform

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt \quad (34)$$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) \exp(i\omega t) d\omega \quad (35)$$

$$2\pi f(-t) = \int_{-\infty}^{\infty} F(\omega) \exp(-i\omega t) d\omega \text{ and hence}$$

$$\text{if } f(t) \longleftrightarrow F(\omega), \text{ then } F(t) \longleftrightarrow 2\pi f(-\omega) \quad (36)$$

(this is the Law of Symmetry).

$$f(at) \longleftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right) : \text{(time scaling)} \quad (37)$$

$$f(t - t_0) \longleftrightarrow F(\omega) \exp(-i\omega t_0) : \text{(time shift)} \quad (38)$$

$$f(t) \exp(i\omega_0 t) \longleftrightarrow F(\omega - \omega_0) : \text{(frequency shift)} \quad (39)$$

$$\frac{d^n f}{dt^n} \longleftrightarrow (i\omega)^n F(\omega) : \text{(time differentiation)}$$

(16)

$$(-it)^n f(t) \longleftrightarrow \frac{d^n F(\omega)}{d\omega^n} : (\text{frequency differentiation})$$

$$m_n = \int_{-\infty}^{\infty} t^n f(t) dt ; (-1)^n m_n = \left[\frac{d^n F(\omega)}{d\omega^n} \right]_{\omega=0} \quad (41) \quad (42)$$

Rectangular pulse:

$$p_T(t) = 1 \quad \text{if } |t| < T, = 0 \text{ elsewhere}$$

$$p_T(\omega) \longleftrightarrow p_T(\omega) \cdot \frac{2 \sin \omega T}{\omega} \quad (43)$$

$$\text{Fourier kernel: } \frac{\sin at}{\pi b} \longleftrightarrow p_a(\omega) \quad (44)$$

$$\text{Triangular pulse: } q_T(t) = 1 - \frac{|t|}{T} \quad \text{if } |t| < T \text{ and}$$

$$= 0 \text{ elsewhere. Its Fourier transform} = \frac{4 \sin^2 \frac{\omega T}{2}}{\omega^2 T} \quad (45)$$

$$\text{Fejer kernel: } \frac{\sin^2 ab}{\pi a b^2} \longleftrightarrow q_{2a}(\omega) \quad (46)$$

$$\exp(-at^2) \longleftrightarrow \sqrt{\frac{\pi}{a}} \exp\left(-\frac{\omega^2}{4a}\right) \quad (a > 0) \quad (47)$$

$$a = \frac{1}{2}. \quad \exp\left(-\frac{t^2}{2}\right) \longleftrightarrow \sqrt{2\pi} \exp\left(-\frac{\omega^2}{2}\right) \quad (48)$$

$$\text{Time convolution: } f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\theta) f_2(t-\theta) d\theta$$

$$\text{and its F.T.} = F_1(\omega) F_2(\omega). \quad (49)$$

$$\text{Frequency convolution: } f_1(t) f_2(t) \longleftrightarrow \frac{1}{2\pi} F_1(\omega) * F_2(\omega) \quad (50)$$

$$\text{Parseval formula: } \int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega \quad (51)$$

$$\int_{-\infty}^{\infty} f_1(t) f_2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \overline{F_1(\omega)} F_2(\omega) d\omega \quad (52)$$

Truncation of $F(\omega)$ above $|\omega| = a$ is achieved

$$Z(n)$$

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45)

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The Fourier expansion (b) is a decomposition of the signal $f(x)$ into harmonic components. It is a case of synthesis of the Fourier components yielding the signal and formulae (2) constitute Fourier analysis of the signal. In the synthesis relation (b), viz.,

we see the decomposition of f into a sum of

mutually orthogonal components $g_n(x) = c_n \exp(inx)$

vectors have one ^{gonal} orthonormal property:

$$\langle g_m, g_n \rangle = \int_{-\pi}^{2\pi} g_m(x) \overline{g_n(x)} dx = 2\pi \delta_{mn} \quad (5)$$

Further, the basic functions $w_n(x) = \exp(inx)$

($n = 0, \pm 1, \pm 2, \dots$) arise from the single

function $w(x) = \exp(ix)$ through integral

dilatations as seen from the relation $w_n(x) = w(nx)$

From (32) (with interval $(0, 2\pi)$ instead of $(-\pi, \pi)$)

we see that

$$\int_0^{2\pi} |f(x)|^2 dx = \pi \sum_{k=0}^{\infty} (|a_k|^2 + |b_k|^2) = 2\pi \sum_{-\infty}^{\infty} |c_k|^2 \quad (57)$$

Let ℓ^2 denote the space of all square-summable

bi-infinite sequences $\{c_n\}$ so that $\sum |c_n|^2 < \infty$.

Choose the square root of $(2\pi)^{-1} \int_0^{2\pi} |f(x)|^2 dx$

as the norm of measurement of functions in $L^2(0, 2\pi)$

and the square root of $\sum_{-\infty}^{\infty} |c_n|^2$ as the norm

of the space of sequences $\{c_n\}$, viz., ℓ^2 . Identity

(57) above shows the 'isometry' of the two spaces

$L^2(0, 2\pi)$ of functions $f(x)$ and ℓ^2 of sequences $\{c_n\}$.

We may therefore assert that: every 2π -periodic

square-integrable function is an ℓ^2 -linear

combination of integral dilatations of the basic function

$W(x) = \exp(ix)$. This is a sinusoid and is

the only function required to generate all 2π -

periodic, square-summable functions. $W_n(x) =$

$W(nx) = \exp(inx)$ has the period $\frac{2\pi}{n}$ and its

(circular) frequency is n . Thus every $L^2(0, 2\pi)$

function is composed of waves with various frequencies.

Let $\psi \in L^2(\mathbb{R}) =$ the space of measurable

functions defined on the real line $\mathbb{R}$ and with the

stipulation $\int_{-\infty}^{\infty} |\psi|^2 dx < \infty$. The functions

$W_n(x)$ seen above do not belong to $L^2(\mathbb{R})$. The

integral shifts $\psi(x-k)$ where $k =$ any integer

will cover $\mathbb{R}$ but do not provide for different

frequencies. Let $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

be the set of all integers. Consider the functions

$$\psi_{j,k}(x) = 2^{j/2} \psi(2^j x - k), \quad j, k \in \mathbb{Z} \quad (58)$$

If $\psi \in L^2(\mathbb{R})$ and the bi-infinite set of functions

$\psi_{j,k}$ defined in (58) have the orthonormal property,

$$\langle \psi_{j,a}, \psi_{k,b} \rangle = \delta_{ja} \delta_{kb} \quad (59)$$

the function ψ is called an orthogonal / orthonormal wavelet and every $f \in L^2(\mathbb{R})$ has then the expansion

$$f(x) = \sum_{j=-\infty}^{\infty} \sum_k c_{j,k} \psi_{j,k}(x) \quad (60)$$

The coefficients $c_{j,k}$ in the above wavelet expansion of $f(x) \in L^2(\mathbb{R})$ are the inner products

$$c_{j,k} = \langle f, \psi_{j,k} \rangle. \quad (61)$$

The convergence of the expansion (60) means that

$$\lim_{M, N \rightarrow \infty} \left\| f - \sum_{j=-M}^{N_2} \sum_{k=-M_1}^{N_1} c_{j,k} \psi_{j,k} \right\|_2 = 0 \quad (62)$$

$\|f\|_2 = (\langle f, f \rangle)^{1/2}$ = the norm for the space $L^2(\mathbb{R})$.

Let W_ψ denote the integral transform defined

for the function $f \in L^2(\mathbb{R})$ through the relation:

$$(W_\psi f)(b, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(x) \overline{\psi\left(\frac{x-b}{a}\right)} dx \quad (63)$$

where $a \neq 0$ and $b \in \mathbb{R}$. The coefficients $c_{j,k}$ in (60)

are then expressible in the form

$$c_{j,k} = (W_\psi f)\left(\frac{x}{2^j}, \frac{1}{2^j}\right) \quad (64)$$

The wavelet series (60) and the wavelet transform (63)

are related through one common relation, viz., (64),

Such an advantage is not forthcoming in Fourier

Series and Fourier transform representations.

The Fourier transform is a mapping of $L^2(\mathbb{R})$ onto $L^2(\mathbb{R})$ (cf. eqns. (34), (35)). If f is an analog signal with finite energy defined by the norm $\|f\|_2 = (\langle f, f \rangle)^{1/2}$ (in the time domain) the Fourier transform $\hat{f} = F = F(\omega) = \hat{f}(\omega)$ is the spectrum of the signal in the frequency domain (allowing for negative frequencies also) and Parseval's identity (51) or (52) is valid for functions $f, g \in L^2(\mathbb{R})$. The energy of the signal f is $\|f\|_2 = \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_2$ and is thus directly proportional to the spectral content. However, to extract the spectral information of the signal, we need information concerning $f(t)$ over the entire infinite time range (cf. (34)). Further we cannot even see the time range $(\mathbb{R})$ that influences the determination of $\hat{f}(\omega)$ for a particular frequency ω . It is desirable to find the time bands that yield

Spectral information on a specified frequency band.
 For high frequency it is desirable to have a small
 time band so as to have better accuracy. Likewise,
 for low frequency we should have a large time
 band to obtain complete information. Thus we
 need a flexible time-frequency window possessing
 the feature of resilience to narrow at high
 centre-frequency band and to widen at low centre-
 frequency band. This is referred to as zoom-in
 and zoom-out capability.

Window function : A nontrivial function

$f \in L^2(\mathbb{R})$ is called a window function if $x f(x)$
 also $\in L^2(\mathbb{R})$. The centre of the window is

$$t^* = \frac{1}{\|f\|_2^2} \int_{-\infty}^{\infty} x |f(x)|^2 dx \quad (65)$$

and its radius is

$$\Delta_f = \frac{1}{\|f\|_2^2} \int_{-\infty}^{\infty} (x - t^*)^2 |f(x)|^2 dx \quad (66)$$

The width of the window is $2 \Delta_f$.

Let ψ and its Fourier transform $\hat{\psi}$ be both

window functions and let their centres be

t^* , ω^* , their radii = $\Delta\psi$ and $\Delta\hat{\psi}$. The

integral wavelet transform of $f(t)$ is

$$(W_\psi f)(t, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt \quad (67)$$

and this localizes the signal $f(t)$ with a time

window $[b + a t^* - a \Delta\psi, b + a t^* + a \Delta\psi]$. This

is time-localization with the centre of the window at

$b + a t^*$ and width = $2a \Delta\psi$. By Parseval's

formula, (65), (67)

$$(W_\psi f)(t, a) = |a|^{-1/2} \langle f(t), \psi\left(\frac{t-b}{a}\right) \rangle = \frac{|a|^{-1/2}}{2\pi} \langle \hat{f}, \widehat{\psi\left(\frac{t-b}{a}\right)} \rangle$$

= $\frac{|a|^{-1/2}}{2\pi}$ x inner product of $\hat{f}(\omega)$ and the Fourier

transform of $\psi\left(\frac{t-b}{a}\right)$; the latter is $a \exp(-ib\omega) \hat{\psi}(a\omega)$.

Introduce the function $\eta(\omega) = \hat{\psi}(\omega + \omega^*)$; then

$$\hat{\psi}(a\omega) = \eta\left(a\left[\omega - \frac{\omega^*}{a}\right]\right) \quad \text{and}$$

$$(W_\psi f)(t, a) = \frac{a |a|^{-1/2}}{2\pi} \int_{-\infty}^{\infty} (\exp(ib\omega) \hat{f}(\omega) \overline{\eta(a(\omega - \frac{\omega^*}{a}))}) d\omega$$

Hence $(W_\psi f)(t, a)$ gives localized information of the

spectrum $\hat{f}(\omega)$ also and the frequency window is

$$\left[\frac{\omega^*}{a} - \frac{1}{a} \Delta\hat{\psi}, \quad \frac{\omega^*}{a} + \frac{1}{a} \Delta\hat{\psi} \right] \quad (68)$$

(24)
 with centre at $\frac{\omega^*}{a}$ and radius = $\frac{\Delta\phi}{a}$. There's
 a multiplier $\frac{a}{2\pi}$ and a linear phase shift $\exp(i\omega t)$
 in the integral in the frequency plane but this
 does not alter the observation that localized information
 of the spectrum $\hat{f}(\omega)$ is contained in $(\omega, t) (t, a)$.

The 'time-frequency window' is determined by
 the time segment $[t + at^* - a\Delta\phi, t + at^* + a\Delta\phi]$
 and the frequency segment $[\frac{\omega^*}{a} - \frac{1}{a}\Delta\phi, \frac{\omega^*}{a} + \frac{1}{a}\Delta\phi]$.

The choice of the wavelet ψ must be such that
 $\omega^* > 0$. The time segment and the frequency
 segment define a rectangle whose area = $4\Delta\phi\Delta\phi$
 and this is a constant. The "constant-Q" of the

$$\text{frequency analysis} = \frac{\omega^*/a}{2\Delta\phi/a} = \frac{\omega^*}{2\Delta\phi} \quad \text{and}$$

this is independent of the location of the ~~centre-frequency~~
 centre-frequency. From the "time-frequency window"

$$[t + at^* - a\Delta\phi, t + at^* + a\Delta\phi] \times \left[\frac{\omega^*}{a} - \frac{\Delta\phi}{a}, \frac{\omega^*}{a} + \frac{\Delta\phi}{a} \right]$$

it is clear that it narrows for large ^{centre-}frequency $\frac{\omega^*}{a}$ and

widens for small centre-frequency $\frac{\omega^*}{a}$.

A function $\psi \in L^2(\mathbb{R})$ is an R-function if

$\psi_{j,k} = 2^{j/2} \psi(2^j x - k)$ is a Riesz basis of $L^2(\mathbb{R})$

{i.e. the linear span of $\psi_{j,k}$ ($j, k \in \mathbb{Z}$) is dense in $L^2(\mathbb{R})$ } and constants A, B exist with

$$0 < A \leq B < \infty \quad \text{and}$$

$$A \|\{c_{j,k}\}\|_{\ell^2}^2 \leq \left\| \sum_{j,k=-\infty}^{\infty} c_{j,k} \psi_{j,k} \right\|_2^2 \leq B \|\{c_{j,k}\}\|_{\ell^2}^2 \quad (69)$$

for all doubly bi-infinite square-summable

sequences $\{c_{j,k}\}$. The last part of the statement

means that $\|\{c_{j,k}\}\|_{\ell^2}^2 = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} |c_{j,k}|^2 < \infty$.

If ψ is an R-function, there is a unique

Riesz basis $\{\psi_{j,k}\}$ of $L^2(\mathbb{R})$ dual to $\{\psi_{j,k}\}$, i.e.

$$\langle \psi_{j,k}, \psi_{l,m} \rangle = \delta_{j,l} \delta_{k,m}, \quad (j,k,l,m) \in \mathbb{Z}.$$

Then, every function $f \in L^2(\mathbb{R})$ has the (unique) series expansion

$$f(x) = \sum_{j,k=-\infty}^{\infty} (\langle f, \psi_{j,k} \rangle) \psi_{j,k}(x). \quad (70)$$

However this may not be a wavelet series since

there may not be a $\tilde{\psi} \in L^2(\mathbb{R})$ such that $\psi_{j,k} = \tilde{\psi}_{j,k}(x)$.

If $\{\psi_{j,k}\}$ is an orthonormal basis of $L^2(\mathbb{R})$ (c.f. (59), (60), (61)), $\tilde{\psi} = \psi$. Whether this is the case or not, the coefficients in the expansion (70) are values of the integral wavelet transform of f relative to ψ .

Every wavelet ψ - orthogonal or not - generates a wavelet series representation of any $f \in L^2(\mathbb{R})$; i.e.

$$f(x) = \sum_{j,k=-\infty}^{\infty} c_{j,k} \psi_{j,k}(x) \quad (71)$$

where each $c_{j,k}$ is the integral wavelet transform of f relative to the dual $\tilde{\psi}$ of ψ and evaluated with $(b, a) = (\frac{x}{2^j}, \frac{1}{2^j})$.

Gabor Transform:

$$\text{let } g_{\alpha}(b) = \frac{1}{2\sqrt{\pi\alpha}} \exp\left(-\frac{b^2}{4\alpha}\right) \quad (\alpha > 0) \quad (72)$$

(Gaussian function). The Gabor transform of

$f \in L^2(\mathbb{R})$ is

$$(G_{\alpha}^b f)(\omega) = \int_{-\infty}^{\infty} (\exp(-i\omega b)) f(b) g_{\alpha}(b-b) db \quad (73)$$

This localizes the Fourier transform of f around $b = b$.

(27)

$$\begin{aligned}
 \int_{-\infty}^{\infty} (G_b^\alpha f)(\omega) d\omega &= \int_{-\infty}^{\infty} d\omega \left(\int_{-\infty}^{\infty} f(t) \exp(-i\omega t) g_\alpha(t-b) dt \right) \\
 &= \int_{-\infty}^{\infty} f(t) \exp(-i\omega t) dt \left(\int_{-\infty}^{\infty} g_\alpha(t-b) d\omega \right) \\
 &= \hat{f}(\omega) \quad \text{since} \quad \int_{-\infty}^{\infty} g_\alpha(t-b) d\omega = 1.
 \end{aligned}$$

The set $\{G_b^\alpha f : b \in \mathbb{R}\}$ of Gabor transforms of f decomposes the Fourier transform $\hat{f}(\omega)$ of f exactly to give its local spectral information.

g_α is an even function and its centre is zero. The radius $\Delta_{g_\alpha} = \sqrt{\alpha}$ and the width of the window

$$\text{is } 2 \Delta_{g_\alpha} = 2\sqrt{\alpha}$$

$$\text{Set } G_{b,\omega}^\alpha(t) = (\exp(i\omega t)) g_\alpha(t-b) \quad (74)$$

$$(G_b^\alpha f)(\omega) = \langle f, G_{b,\omega}^\alpha \rangle = \int_{-\infty}^{\infty} f(t) \overline{G_{b,\omega}^\alpha(t)} dt$$

While $G_b^\alpha f$ is the localization of the Fourier transform of f , it also windows the signal $f(t)$.

$$\hat{G}_{b,\omega}^\alpha(\eta) = (\exp[-ib(\eta-\omega)]) \exp[-\alpha(\eta-\omega)^2] \quad (75)$$

$$(G_b^\alpha f)(\omega) = \langle f, G_{b,\omega}^\alpha \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{G}_{b,\omega}^\alpha \rangle$$

$$\begin{aligned}
 &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\eta) (\exp[ib(\eta-\omega)]) \exp[-\alpha(\eta-\omega)^2] d\eta \\
 &= \frac{e^{-ib\omega}}{2\sqrt{\pi\alpha}} \left(G_{\frac{1}{4\alpha}}^{\frac{1}{4\alpha}} \hat{f} \right)(-b). \quad (76)
 \end{aligned}$$

$$\int_{-\infty}^{\infty} e^{-i\omega b} f(t) g_\alpha(t-b) dt = \text{F.T. of } f(t) g_\alpha(t-b)$$

(28)

= window F.T. of f with window function g_α at $t=t$

The right side of (76) equals $\left(\sqrt{\frac{\pi}{\alpha}} e^{-i\omega t}\right) \times$ (window

inverse F.T. of $\hat{f}$ with the window function

$g_{\frac{1}{4\alpha}}$ at $\eta = \omega$.

The products of the widths

of these two windows is

$$(2 \Delta_{g_\alpha})(2 \Delta_{g_{\frac{1}{4\alpha}}}) = 2. \quad (77)$$

Consider $H_{b,w}^\alpha(\eta) = \frac{1}{2\pi} \widehat{G_{b,w}^\alpha}(\eta) = \frac{e^{i\eta b}}{2\sqrt{\pi\alpha}} e^{-i\eta w} g_{\frac{1}{4\alpha}}(\eta - \omega).$

Then $\langle f, G_{b,w}^\alpha \rangle = \langle \hat{f}, H_{b,w}^\alpha \rangle. \quad (78)$

Thus, the information obtained by investigating an

analog signal $f(t)$ at $t=t$ using the window

function $G_{b,w}^\alpha$ is also obtained by observing the

spectrum $\hat{f}(\eta)$ of the signal in a neighbourhood

of the frequency $\eta = \omega$ by using the window function

$H_{b,w}^\alpha$. The product of the widths of the window

functions is

$$(2 \Delta_{G_{b,w}^\alpha})(2 \Delta_{H_{b,w}^\alpha}) = (2 \Delta_{g_\alpha})(2 \Delta_{g_{\frac{1}{4\alpha}}}) = 2. \quad (79)$$

The cartesian product of the two windows is

$$[t - \sqrt{\alpha}, t + \sqrt{\alpha}] \times \left[\omega - \frac{1}{2\sqrt{\alpha}}, \omega + \frac{1}{2\sqrt{\alpha}} \right] \quad (80)$$

which is a rectangular time-frequency window.

The time-window has width $2\sqrt{\alpha}$ = width of the time-frequency window. The frequency window has width $\frac{1}{\sqrt{\alpha}}$ and equals the height of the time-frequency window. The width of the time-frequency window is unchanged whatever be the frequency of the spectrum.

The Gabor transform is a window Fourier transform with a Gaussian g_α as the window function. The Fourier transform $\hat{g}_\alpha$ is also a Gaussian and both g_α and $\hat{g}_\alpha$ can be used for time-frequency localization.

Short-time Fourier Transform (STFT):

If $w(t) \in L^2(\mathbb{R})$ and both w and its F.T.

$\hat{w}$ satisfy the ~~stationary~~ conditions that

1) $w(t) \in L^2(\mathbb{R})$ and $\omega \hat{w}(\omega) \in L^2(\mathbb{R})$, then the

window Fourier transform

$$(\tilde{g}_t f)(\omega) = \int_{-\infty}^{\infty} \exp(-i\omega t) f(t) \overline{w(t-\omega)} dt \quad (81)$$

is called a "Short-time Fourier Transform" (STFT).

Here both w and $\hat{w}$ satisfy conditions for being window functions and they must be continuous functions. Obviously Gaussian functions can be used to define STFT.

B-spline functions. The first order B-spline $N_1(t)$

is $N_1(t) = 1$ if $0 \leq t < 1$ and 0 otherwise. The

m th order cardinal B-spline is defined recursively by integral convolution.

$$\begin{aligned} N_m(x) &= \int_{-\infty}^{\infty} N_{m-1}(x-t) N_1(t) dt \quad (m \geq 2) \\ &= \int_0^1 N_{m-1}(x-t) dt \\ \hat{N}_m(\omega) &= \left(\frac{1 - \exp(-i\omega)}{i\omega} \right)^m = \left(\exp(-im\omega/2) \right) \left(\frac{\sin \frac{\omega}{2}}{\omega/2} \right)^m \end{aligned} \quad (82)$$

Both N_m and $\hat{N}_m$ satisfy the criteria for being window functions.

If w and $\hat{w}$ are both window functions with widths Δ_w and $\Delta_{\hat{w}}$, we have by the uncertainty relation

$$\Delta_w \Delta_{\hat{w}} \geq \frac{1}{2}$$

and equality holds only if $w(t) = A \exp(i\alpha t) g_{\alpha}(t-t_0)$

(31)

($A \neq 0$, $\alpha > 0$, $a, b \in \mathbb{R}$). The time-frequency window area $= 4 \Delta_w \Delta_\omega > 2$ and equality holds only for the Gabor transform. The Gabor transform is the STFT with the smallest time-frequency window.

Let $w \in L^2(\mathbb{R})$ be chosen ^{such} that $\|w\|_2 = 1$ and let both w and $\hat{w}$ satisfy the window function criterion so that $bw(t) \in L^2(\mathbb{R})$ and $w\hat{w}(\omega) \in L^2(\mathbb{R})$. Then, for any $f, g \in L^2(\mathbb{R})$ we can show that

$$2\pi \langle f, g \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle f, w_{b,w} \rangle \overline{\langle g, w_{b,w} \rangle} db d\omega \quad (84)$$

By the choice of g to be the Gaussian and taking limits as $\alpha \rightarrow 0+$, we see that for every $f \in L^2(\mathbb{R})$, at every point x of continuity of f , we have

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\exp(i\omega x) (\tilde{G}_{b,w}(f)(\omega))] w(x-b) d\omega db. \quad (85)$$

In the STFT (81) we have the window area $= 4 \Delta_w \Delta_\omega$ and the time-frequency window, viz.,

$$[b^* + b - \Delta_w, b^* + b + \Delta_w] \times [\omega^* + \omega - \Delta_\omega, \omega^* + \omega + \Delta_\omega] \quad (86)$$

is rigid with no (δ^2) change in width in observing any frequency band (on octave) $[\omega^* + \omega - \Delta\omega, \omega^* + \omega + \Delta\omega]$ centered at $\omega^* + \omega$. Frequency & the number of cycles per unit time and to locate high frequency phenomena we must have a narrow time window; to analyze low frequency behaviour, we must have a wide time-window. Thus STFT is unsuitable for analyzing signals with both very high and very low frequencies. It is desirable to have a flexible time-frequency window that narrows when observing high-frequency phenomena and widens when studying low-frequency environments.

Basic Wavelet. If $\psi \in L^2(\mathbb{R})$ and

$$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega < \infty \quad (87)$$

ψ is called a basic wavelet. Further, if both ψ

and $\hat{\psi}$ are such that $t\psi(t) \in L^2(\mathbb{R})$, $\omega\hat{\psi}(\omega) \in L^2(\mathbb{R})$

the basic wavelet ψ provides a time-frequency window

with area $= 4\Delta_\psi\Delta_\phi$ (which is finite). It follows that

$\hat{\psi} = \int_{-\infty}^{\infty} e^{-i\omega t} \psi(t) dt$ is continuous and the

finiteness of c_{ψ} in (87) mandates that $\hat{\psi}(0) = 0$. Thus

$$\hat{\psi}(0) = \int_{-\infty}^{\infty} \psi(t) dt = 0 \quad \text{and the area under } \psi(t)$$

is therefore small, justifying the ^{basic} ~~appellation~~ ^{appellation} 'wavelet'.

Relative to every ^{basic} wavelet ψ we define on $L^2(\mathbb{R})$

the integral wavelet transform (IWT) in the form

$$(W_{\psi} f)(b, a) = |a|^{-1/2} \int_{-\infty}^{\infty} f(t) \overline{\psi\left(\frac{t-b}{a}\right)} dt, \quad f \in L^2(\mathbb{R}) \quad (88)$$

with $a, b \in \mathbb{R}$ and $a \neq 0$. We see from (88) that

$$(W_{\psi} f)(b, a) = \langle f, \psi_{b;a} \rangle \quad (89)$$

where $\psi_{b;a}(t) = |a|^{-1/2} \psi\left(\frac{t-b}{a}\right)$. The centre and

radius of ψ are b^* and Δ_{ψ} and $\psi_{b;a}(t)$ is a

is a window function with centre at $b + at^*$ and

radius $= a \Delta_{\psi}$. The IWT (89) gives local

information of an analog signal $f(t)$ with a time-window

$$[b + at^* - a\Delta_{\psi}, b + at^* + a\Delta_{\psi}] \quad (90)$$

which narrows for small values of $|a|$ and widens

when $|a|$ is large.

$$\frac{1}{2\pi} \hat{\psi}_{b;a}(\omega) = \frac{|a|^{-1/2}}{2-\pi} \int_{-\infty}^{\infty} (e^{-i\omega t}) \psi\left(\frac{t-b}{a}\right) dt$$

(34)

$$= \frac{a|a|^{-1/2}}{2\pi} \exp(-i\omega) \hat{\psi}(a\omega) \quad (91)$$

If the window function $\hat{\psi}$ has centre ω^* and radius $\Delta\hat{\psi}$, and $\eta(\omega) = \hat{\psi}(\omega + \omega^*)$, we have using Parseval formula,

$$\begin{aligned} (W_{\psi} f)(t, a) &= \langle f, \psi_{t,a} \rangle = \frac{1}{2\pi} \langle \hat{f}, \hat{\psi}_{t,a} \rangle \\ &= \frac{a|a|^{-1/2}}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\omega) \exp(i\omega t) \eta(a(\omega - \frac{\omega^*}{a})) d\omega. \quad (92) \end{aligned}$$

The window function $\eta(a\omega - \omega^*) = \psi(a\omega)$ has

radius $\frac{1}{a} \Delta\hat{\psi}$. From (92) we see that except for

the multiple $\frac{a|a|^{-1/2}}{2\pi}$ and a linear phase shift of $\exp(i\omega t)$

the IWT $W_{\psi} f$ gives local information of

$\hat{f}$ with a frequency-window $\left[\frac{\omega^*}{a} - \frac{1}{a} \Delta\hat{\psi}, \frac{\omega^*}{a} + \frac{1}{a} \Delta\hat{\psi} \right]$.

We may presume that ω^* and a are positive.

If the frequency variable is identified as constant

multiple of a^{-1} , an adaptive bandpass filter with

pass-band = $\left[\frac{\omega^*}{a} - \frac{1}{a} \Delta\hat{\psi}, \frac{\omega^*}{a} + \frac{1}{a} \Delta\hat{\psi} \right]$ has the

property that the ratio $\frac{\text{centre-frequency}}{\text{bandwidth}}$ is independent

of the location of the centre frequency ($= \frac{\omega^*}{2\Delta\hat{\psi}}$

which is independent of a).

(35)
This is called "constant - a filtering". We

have the rectangular time-frequency window

$$[b + at^* - a\Delta\varphi, b + at^* + a\Delta\varphi] \times \left[\frac{\omega^*}{a} - \frac{1}{a}\Delta\hat{\varphi}, \frac{\omega^*}{a} + \frac{1}{a}\Delta\hat{\varphi} \right] \quad (93)$$

in the $t-\omega$ plane with width $2a\Delta\varphi$ and height $\frac{2}{a}\Delta\hat{\varphi}$.

The time-window narrows ($a > 0$ and small) for large frequency phenomena and widens ($a > 0$ and large) for low-frequency phenomena.

If ψ is a basic wavelet defining the IWT W_ψ and $f, g \in L^2(\mathbb{R})$, ~~then at points of~~ ^{we have}

~~continuity of f (on $\mathbb{R}$)~~

$$c_\psi(\langle f, g \rangle) = \int_{-\infty}^{\infty} \int (W_\psi f)(b, a) \overline{(W_\psi g)(b, a)} \frac{da db}{a^2} \quad (94)$$

with c_ψ defined in (87). Further, at all

points $x \in \mathbb{R}$ where f is continuous,

$$f(x) = \frac{1}{c_\psi} \int_{-\infty}^{\infty} \int [(W_\psi f)(b, a)] \psi_{b;a}(x) \frac{da db}{a^2}. \quad (95)$$

Dyadic wavelets: Let ψ be a basic wavelet and $\hat{\psi}$

its Fourier transform. We partition the infinite

interval $(0, \infty)$ in the form

$$\bigcup_{j=-\infty}^{\infty} (2^j \Delta\hat{\varphi}, 2^{j+1} \Delta\hat{\varphi}] \quad (96)$$

(36)

where $\Delta \hat{\psi}$ is the radius of the F.T. $\hat{\psi}$, which is assumed to be such that $\omega \hat{\psi}(\omega) \in L^2(\mathbb{R})$. Choice of $\omega^* = 3\Delta \hat{\psi}$ and $a_j = 2^{-j}$ for all $j \in \mathbb{Z}$ implies

that

$$\omega_j = \frac{\omega^*}{a_j} = \frac{3\Delta \hat{\psi}}{a_j} = (3)(2^j)\Delta \hat{\psi} \quad (97)$$

$$\text{and } \left(\frac{\omega^*}{a_j} - \frac{1}{a_j} \Delta \hat{\psi}, \frac{\omega^*}{a_j} + \frac{1}{a_j} \Delta \hat{\psi} \right) = (2^{j+1}\Delta \hat{\psi}, 2^{j+2}\Delta \hat{\psi}) \quad (98)$$

We have then a partition of the positive

frequency domain $(0, \infty)$.

A function $\psi \in L^2(\mathbb{R})$ is called a dyadic wavelet if there exist two positive constants A and B

with $0 < A \leq B < \infty$ such that

$$A \leq \sum_{j=-\infty}^{\infty} |\hat{\psi}(2^j \omega)|^2 \leq B \quad \text{a.e.} \quad (99)$$

This is often called the 'stability' criterion.

If ψ satisfies the stability criterion (99), it

is a basic wavelet satisfying

$$A \ln 2 \leq \int_0^{\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega, \quad \int_0^{\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega} d\omega \leq B \ln 2$$

$$\text{If } A = B, \quad C = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} d\omega = 2A \ln 2.$$

x

x

x

x

x